

THE CORONA THEOREM AND BASS STABLE RANK FOR $M(D(\sum_{i=1}^k a_i \delta_{\zeta_i}))$

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ABSTRACT. In this paper, we prove the corona theorem for $M(D(\mu_k))$ in two different ways, where $\mu_k = \sum_{i=1}^k a_i \delta_{\zeta_i}$. Then we prove that the Bass stable rank of $M(D(\mu_k))$ is one.

1. INTRODUCTION

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc. Let μ be a nonnegative Borel measure on the boundary \mathbb{T} of the unit disc. Let φ_μ be the harmonic function

$$\varphi_\mu(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta).$$

The Dirichlet type space $D(\mu)$ is defined as the space of all analytic functions on \mathbb{D} such that

$$\int_{\mathbb{D}} |f'(z)|^2 \varphi_\mu(z) dA(z)$$

is finite. For any $f \in D(\mu)$, $\|f\|_{D(\mu)}^2 := \|f\|_{H^2(\mathbb{D})}^2 + \int_{\mathbb{D}} |f'(z)|^2 \varphi_\mu(z) dA(z)$.

When $\mu = \frac{dt}{2\pi}$, $D(\frac{dt}{2\pi})$ is the Dirichlet space D .

Dirichlet type spaces were introduced by Richter in [5] when studying analytic two-isometries. In [6], Richter and Sundberg showed that if $f \in D(\delta_\zeta)$, then

$$D_\zeta(f) = \int_{\mathbb{D}} |f'(z)|^2 \frac{1 - |z|^2}{|\zeta - z|^2} dA(z), \quad \zeta \in \mathbb{T}$$

which is a convenient tool in studying these spaces, where $D_\zeta(f) := \|\frac{f-f(\zeta)}{z-\zeta}\|_{H^2(\mathbb{D})}^2$ is called the local Dirichlet integral of f at ζ . Thus,

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for any $f \in D(\mu)$, $\|f\|_{D(\mu)}^2 = \|f\|_{H^2(\mathbb{D})}^2 + \int_{\mathbb{T}} D_{\zeta}(f) d\mu(\zeta) = \|f\|_{H^2(\mathbb{D})}^2 + \int_{\mathbb{T}} \left\| \frac{f-f(\zeta)}{z-\zeta} \right\|_{H^2(\mathbb{D})}^2 d\mu(\zeta)$.

In this paper, we will consider $\mu = \sum_{i=1}^k a_i \delta_{\zeta_i} := \mu_k$, where a_i 's are positive numbers, ζ_i 's are in \mathbb{T} . Let $M(D(\mu_k))$ be the space of multipliers of $D(\mu_k)$, that is

$$M(D(\mu_k)) = \{\phi \in D(\mu_k) : \phi f \in D(\mu_k), \forall f \in D(\mu_k)\}.$$

Also we will consider $D_{l^2}(\mu_k)$, or $\oplus_1^{\infty} D(\mu_k)$, which can be considered as l^2 -valued $D(\mu_k)$ space.

Given $\{\varphi_j\}_{j=1}^{\infty} \subseteq M(D(\mu_k))$, we let $\Phi(z) = (\varphi_1(z), \varphi_2(z), \dots)$. We use M_{Φ} to denote the (column) operator from $D(\mu_k)$ to $\oplus_1^{\infty} D(\mu_k)$ defined by

$$M_{\Phi}(f) = \{\varphi_j f\}_{j=1}^{\infty} \quad \text{for } f \in D(\mu_k).$$

The famous corona theorem goes back to Lennart Carleson. In 1962 Carleson [2] proved the absence of a corona in the maximal ideal space of $H^{\infty}(\mathbb{D})$ by showing that if $\{\varphi_1, \dots, \varphi_n\}$ is a finite set of functions in $H^{\infty}(\mathbb{D})$ satisfying

$$(1.1) \quad \sum_{j=1}^n |\varphi_j(z)|^2 \geq \eta > 0, \quad z \in \mathbb{D}, \quad (\text{Corona condition}).$$

then there are functions $\{f_1, \dots, f_n\} \subseteq H^{\infty}(\mathbb{D})$ with

$$(1.2) \quad \sum_{j=1}^n f_j(z) \varphi_j(z) = 1, \quad z \in \mathbb{D}, \quad (\text{Bezout equation}).$$

This is also equivalent to say that the unit disc is dense in the maximal ideal space of $H^{\infty}(\mathbb{D})$ in the weak* topology. Then it was shown that the corona theorem is also true in $M(D)$, the multiplier of the Dirichlet space D (see Tolokonnikov [10], Xiao [15]). In this paper, we wish to prove the corona theorem for $M(D(\mu_k))$ in two ways. The first version is as follows:

Theorem 1.1. *The set of multiplicative linear functionals consisting of evaluations at points of \mathbb{D} is dense in the maximal ideal space of $M(D(\mu_k))$.*

By the standard Gelfand theory of Banach algebras Theorem 1.1 implies:

Corollary 1.2. *The following are equivalent:*

(i) $\varphi_1, \dots, \varphi_n \in M(D(\mu_k))$ and there exists a $\eta > 0$ such that

$$\sum_{j=1}^n |\varphi_j(z)|^2 \geq \eta > 0, \quad z \in \mathbb{D}.$$

(ii) There are functions $b_1, \dots, b_n \in M(D(\mu_k))$ such that

$$\sum_{j=1}^n \varphi_j(z) b_j(z) = 1, \quad z \in \mathbb{D}.$$

Also the corona theorem has been generalized to infinitely many functions in $H^\infty(\mathbb{D})$ and $M(D)$ (see Rosenblum [7], Tolokonnikov [10] and Trent [13]). The infinite version, given by Rosenblum [7] and Tolokonnikov [10], can be formulate as follows (see Trent [14]):

Theorem 1.3. *Let $\{\varphi_j\}_{j=1}^\infty \subseteq H^\infty(\mathbb{D})$. Suppose that*

$$0 < \epsilon^2 \leq \sum_{j=1}^\infty |\varphi_j(z)|^2 \leq 1, \quad \text{for all } z \in \mathbb{D}.$$

Then there exists $\{e_j\}_{j=1}^\infty \subseteq H^\infty(\mathbb{D})$ such that $\sum_{j=1}^\infty \varphi_j e_j = 1$ and $\sup_{z \in \mathbb{D}} \sum_{j=1}^\infty |e_j(z)|^2 \leq \frac{C_0}{\epsilon^2} \ln \frac{1}{\epsilon^2}$, where C_0 is a constant.

Note that the pointwise hypothesis $\sum_{j=1}^\infty |\varphi_j(z)|^2 \leq 1$ implies that the operator T_Φ defined on $H^2(\mathbb{D})$ in analogy to that of M_Φ is bounded and $\|T_\Phi\| = \sup_{z \in \mathbb{D}} (\sum_{j=1}^\infty |\varphi_j(z)|^2)^{\frac{1}{2}}$. Note that since $M(D(\mu_k)) = D(\mu_k) \cap H^\infty(\mathbb{D})$, the pointwise upper bound hypothesis will not be sufficient to conclude that M_Φ is bounded from $D(\mu_k)$ to $\oplus_1^\infty D(\mu_k)$. Thus, we will replace the assumption $\sum_{j=1}^\infty |\varphi_j(z)|^2 \leq 1$ for $z \in \mathbb{D}$ by the condition $\|M_\Phi\| \leq 1$. Then we have the following theorem:

Theorem 1.4. *Let $\{\varphi_j\}_{j=1}^\infty \subseteq M(D(\mu_k))$. Suppose that*

$$\|M_\Phi\| \leq 1 \quad \text{and} \quad 0 < \epsilon^2 \leq \sum_{j=1}^\infty |\varphi_j(z)|^2 \quad \text{for all } z \in \mathbb{D}.$$

Then there exists $\{b_j\}_{j=1}^\infty \subseteq M(D(\mu_k))$ such that

- (i) $\Phi(z)B(z)^\top = 1$ for all $z \in \mathbb{D}$, and
- (ii) $\|M_B\| \leq \frac{1}{\epsilon} \left(2 + 16\|M_{B_{k-1}}\|^2\right)^{1/2}$, where B_{k-1} is the solution for the corona theorem in $M(D(\mu_{k-1}))$.

We will use induction to prove Theorem 1.1 and Theorem 1.4. In section 4, we show that the Bass stable rank of $M(D(\mu_k))$ is one. Throughout this paper, we use C, C_1, C_2, \dots for absolute constants.

2. CORONA THEOREM FOR $M(D(\mu_k))$

2.1. First, we consider that $k = 1$ and $\mu_k = \delta_1$, the unit point mass at 1. To prove the corona theorem for $M(D(\delta_1))$, we need the following two Lemmas (see [6]).

Lemma 2.1. *Let $f \in D(\delta_1)$. Then*

- (i) $f = f(1) + (z - 1)g$ for some $g \in H^2(\mathbb{D})$ and $D_1(f) = \|g\|_{H^2(\mathbb{D})}^2$.
- (ii) $\lim_{r \rightarrow 1^-} f(r) = f(1)$.
- (iii) $|f(1)| \leq C\|f\|_{D(\delta_1)}$ (see [11]).

Lemma 2.2. *Let $\varphi \in H^\infty(\mathbb{D})$ and $f \in D(\delta_\zeta)$. Then $\varphi f \in D(\delta_\zeta)$ if and only if $f(\zeta) = 0$ or $\varphi \in D(\delta_\zeta)$. Furthermore,*

$$D_\zeta(\varphi f) \leq 2(\|\varphi\|_\infty^2 D_\zeta(f) + |f(\zeta)|^2 D_\zeta(\varphi))$$

and

$$|f(\zeta)|^2 D_\zeta(\varphi) \leq 2(\|\varphi\|_\infty^2 D_\zeta(f) + D_\zeta(\varphi f)).$$

If $f(\zeta) = 0$ then one even has $D_\zeta(\varphi f) \leq \|\varphi\|_\infty^2 D_\zeta(f)$, while the second inequality can be replaced with the trivial observation that the right-hand side is nonnegative.

Thus, by Lemma 2.2, we have $M(D(\mu_k)) = D(\mu_k) \cap H^\infty(\mathbb{D})$, where $\mu_k = \sum_{i=1}^k a_i \delta_{\zeta_i}$. The norm in $D(\mu_k) \cap H^\infty(\mathbb{D})$ is defined by

$$\|f\|_{D(\mu_k) \cap H^\infty(\mathbb{D})} = \|f\|_{D(\mu_k)} + \|f\|_\infty, \quad f \in D(\mu_k) \cap H^\infty(\mathbb{D}).$$

We will use a similar idea as in Lemma 2.1 of [4] to prove the corona theorem for $M(D(\delta_1))$.

For ease of notation, we let $K := M(D(\delta_1)) = D(\delta_1) \cap H^\infty(\mathbb{D})$, and $K_0 := \{f \in K, f(1) = 0\}$. Note that $K_0 \subset K$, and K_0 is a Banach algebra without identity.

Note that evaluation at $z \in \mathbb{D} \cup \{1\}$ is a multiplicative linear functional on K_0 (if $z = 1$, then it is a trivial one). We have the following lemma.

Lemma 2.3. *The set of multiplicative linear functionals consisting of evaluations at points of \mathbb{D} is dense in the set of all multiplicative linear functionals on K_0 .*

Proof. Let m be a non-zero multiplicative linear functional on K_0 , then there exists a function $g_0 \in K_0$, such that $m(g_0) \neq 0$.

If $f \in H^\infty(\mathbb{D})$, define $M(f) := \frac{m(fg_0)}{m(g_0)}$.

Claim: M is well-defined, and M is a non-zero multiplicative linear functional on $H^\infty(\mathbb{D})$.

If we assume that the claim holds, then by Carleson's corona Theorem, there exists a net $(\beta_i)_{i \in I}$ of point evaluations in \mathbb{D} that converges

to M in the weak* topology of the maximal ideal space of $H^\infty(\mathbb{D})$. Note that m is the restriction of M to K_0 :

$$M(f) = \frac{m(fg_0)}{m(g_0)} = \frac{m(f)m(g_0)}{m(g_0)} = m(f), f \in K_0.$$

Also the restriction of $(\beta_i)_{i \in I}$ gives a net of point evaluations in \mathbb{D} that converges to m in the weak* topology on the dual space of K_0 .

We are left to prove the claim: $f \in H^\infty(\mathbb{D})$, $g_0 \in K_0$, so $fg_0 \in K$ by Lemma 2.2. Also $(fg_0)(1) = 0$, so $fg_0 \in K_0$, which implies M is well-defined.

Clearly M is linear, when $f \in H^\infty(\mathbb{D})$,

$$\begin{aligned} |M(f)| &= \left| \frac{m(fg_0)}{m(g_0)} \right| \leq \frac{\|fg_0\|_K}{|m(g_0)|} \\ &= \frac{\|fg_0\|_\infty + \|fg_0\|_{D(\delta_1)}}{|m(g_0)|} \leq \frac{\|f\|_\infty \|g_0\|_\infty + \|f\|_\infty \|g_0\|_{D(\delta_1)}}{|m(g_0)|} \\ &= \frac{\|g_0\|_K}{|m(g_0)|} \|f\|_\infty, \end{aligned}$$

so M is a bounded functional on $H^\infty(\mathbb{D})$.

When $f, h \in H^\infty(\mathbb{D})$, $m(fhg_0)m(g_0) = m(fhg_0g_0) = m(fg_0)m(hg_0)$, thus we get

$$\begin{aligned} M(fh) &= \frac{m(fhg_0)}{m(g_0)} \\ &= \frac{[m(fg_0)m(hg_0)]/m(g_0)}{m(g_0)} \\ &= M(f)M(h). \end{aligned}$$

Therefore the claim is proved. ■

Now, we can prove the following Theorem.

Theorem 2.4. *The set of multiplicative linear functionals consisting of evaluations at points of $\mathbb{D} \cup \{1\}$ is dense in the maximal ideal space of K .*

Proof. Suppose M is a non-zero multiplicative linear functional on K .

Let $m = M|_{K_0}$, then m is a multiplicative linear functional on K_0 . If $f \in K$, then $f - f(1) \in K_0$, so $M(f) = f(1) + m(f - f(1))$.

Case 1. If $m = 0$, then $M(f) = f(1)$, so M is the point evaluation at 1.

Case 2. If $m \neq 0$, then by Lemma 2.3, there exists a net $(\beta_i)_{i \in I}$ of point evaluations in \mathbb{D} that converges to m in the weak* topology on

the dual space of K_0 . Therefore, for all $f \in K$,

$$\begin{aligned} M(f) &= f(1) + m(f - f(1)) = f(1) + (\lim_{i \in I} \beta_i)(f - f(1)) \\ &= f(1) + \lim_{i \in I} (f(\beta_i) - f(1)) \\ &= \lim_{i \in I} f(\beta_i) = (\lim_{i \in I} \beta_i)(f). \end{aligned}$$

Thus $M = \lim_{i \in I} \beta_i$, and this completes the proof. \blacksquare

Remark 2.5. For any $f \in K$, $0 < r < 1$, let $E_r(f) = f(r)$, then from Lemma 2.1 we have $f(r) \rightarrow f(1)$ as $r \rightarrow 1$. Thus $E_r \rightarrow E_1$ in the weak star topology of K as $r \rightarrow 1$, which implies the set of multiplicative linear functionals consisting of evaluations at points of \mathbb{D} is dense in the maximal ideal space of K .

2.2. In this subsection, we consider general $k \geq 1$. Let μ be a Borel measure in \mathbb{T} with $\mu(\zeta) = 0$, where $\zeta \in \mathbb{T}$, and suppose that \mathbb{D} is dense in the maximal ideal space of $M(D(\mu))$. Let $H := M(D(\mu)) \cap D(\delta_\zeta)$ and $H_0 := \{f \in H, f(\zeta) = 0\}$. Then we have:

Lemma 2.6. H is a Banach algebra, $H_0 \subset H$ and H_0 is a Banach algebra without identity.

Proof. We only need to verify that H is an algebra. Suppose $f, g \in H = M(D(\mu)) \cap D(\delta_\zeta)$, then $fg \in M(D(\mu))$. Also $f - f(\zeta) \in H$ implies $\frac{f-f(\zeta)}{z-\zeta}g \in H^2(\mathbb{D})$, thus

$$fg = (z - \zeta) \left(\frac{f - f(\zeta)}{z - \zeta} g \right) + f(\zeta)g \in D(\delta_\zeta),$$

and so $fg \in H$. \blacksquare

Lemma 2.7. The set of multiplicative linear functionals consisting of evaluations at points of \mathbb{D} is dense in the maximal ideal space of H_0 .

Proof. Let m be a non-zero multiplicative linear functional on H_0 , then there exists a function $g_0 \in H_0$, such that $m(g_0) \neq 0$.

If $f \in M(D(\mu))$, define $M(f) := \frac{m(fg_0)}{m(g_0)}$.

Claim: M is well-defined, and M is a non-zero multiplicative linear functional on $M(D(\mu))$.

The proof of the claim is similar to the argument in Lemma 2.3. Then there exists a net $(\beta_i)_{i \in I}$ of point evaluations in \mathbb{D} that converges to M in the Gelfand topology of the maximal ideal space of $M(D(\mu))$. Note that m is the restriction of M to H_0 . Also the restriction of $(\beta_i)_{i \in I}$ gives a net of point evaluations in \mathbb{D} that converges to m in the weak* topology on the dual of H_0 . \blacksquare

By the same argument as in Theorem 2.4 we have the following Proposition:

Proposition 2.8. *The set of multiplicative linear functionals consisting of evaluations at points of \mathbb{D} is dense in the maximal ideal space of H .*

Now we can prove Theorem 1.1.

Proof. This clearly follows from Proposition 2.8 and induction. \blacksquare

Remark 2.9. *If we let $d\mu = \frac{dt}{2\pi}$, then $D(\frac{dt}{2\pi})$ is the Dirichlet space D . By Tolokonnikov [10], Xiao [15] we have the corona theorem in $M(D)$, then by Proposition 2.8 we also have the corona theorem in $M(D) \cap D(\delta_\zeta)$ for any $\zeta \in \mathbb{T}$.*

3. INFINITE VERSION FOR $M(D(\mu_k))$

3.1. First, we consider $M(D(\delta_1))$.

The following Lemma can be derived from [13, Lemma 6] (see also [8]).

Lemma 3.1. *Let $\{a_j\}_{j=1}^\infty \in l^2$ and $A = (a_1, a_2, \dots) \in B(l^2, \mathbb{C})$. Then there exists an $\infty \times \infty$ matrix Q_A , such that the entries of Q_A belong to the set $\{0, \pm a_j : j = 1, 2, \dots\}$ and Q_A satisfies*

- (a) *range of $Q_A \subseteq \text{kernel of } A$.*
- (b) *$(AA^*)I - A^*A = Q_A Q_A^*$.*
- (c) *If $\{d_j\}_{j=1}^\infty \in l^2$ and $D = (d_1, d_2, \dots)$, then*

$$(AD^\top)I - D^\top A = Q_A Q_D^\top.$$

We need one lemma before we prove the corona theorem for infinitely many functions in $M(D(\delta_1))$.

Lemma 3.2. *Let $\{\varphi_j\}_{j=1}^\infty \subseteq M(D(\delta_1))$. Then*

- (i) *M_Φ is a bounded operator if and only if $\sum_{j=1}^\infty \|\varphi_j\|_{D(\delta_1)}^2$ and $\sup_{z \in \mathbb{D}} \sum_{j=1}^\infty |\varphi_j(z)|^2$ are finite.*
- (ii) *If $\|M_\Phi\| \leq 1$ and $0 < \epsilon^2 \leq \sum_{j=1}^\infty |\varphi_j(z)|^2$ for all $z \in \mathbb{D}$, then*

$$\Phi(1) = (\varphi_1(1), \varphi_2(1), \dots) \neq 0.$$

- (iii) *If $\|M_\Phi\| \leq 1$ and $f = \sum_{i=1}^\infty [\varphi_i - \varphi_i(1)] \overline{\varphi_i(1)}$, then $f \in M(D(\delta_1))$ and $f(1) = 0$.*

Proof. (i): Suppose that M_Φ is bounded from $D(\delta_1)$ to $\oplus_1^\infty D(\delta_1)$ with $\|M_\Phi\| \leq 1$, then $\sup_{z \in \mathbb{D}} \sum_{j=1}^\infty |\varphi_j(z)|^2 \leq 1$ (see [13]). Let $f = 1 \in D(\delta_1)$, then

$$\begin{aligned} \sum_{j=1}^\infty \|\varphi_j\|_{D(\delta_1)}^2 &= \|M_\Phi f\|_{\oplus_1^\infty D(\delta_1)}^2 \\ &\leq \|M_\Phi\|^2 \|1\|_{D(\delta_1)}^2 \leq 1. \end{aligned}$$

Conversely suppose $\sup_{z \in \mathbb{D}} \sum_{j=1}^\infty |\varphi_j(z)|^2 \leq 1$ and $\sum_{j=1}^\infty \|\varphi_j\|_{D(\delta_1)}^2 \leq 1$. Let $f \in D(\delta_1)$, suppose $f = f(1) + (z-1)g$ for some $g \in H^2(\mathbb{D})$, then $D_1(f) = \|g\|_{H^2(\mathbb{D})}^2$ and

$$\begin{aligned} \|M_\Phi f\|_{\oplus_1^\infty D(\delta_1)}^2 &= \sum_{j=1}^\infty \|\varphi_j f\|_{D(\delta_1)}^2 \\ &= \sum_{j=1}^\infty \|\varphi_j f\|_{H^2(\mathbb{D})}^2 + \sum_{j=1}^\infty \left\| \frac{\varphi_j f - (\varphi_j f)(1)}{z-1} \right\|_{H^2(\mathbb{D})}^2 \\ &\leq \|f\|_{H^2(\mathbb{D})}^2 + \sum_{j=1}^\infty \left[2 \left\| \frac{\varphi_j f(1) - (\varphi_j f)(1)}{z-1} \right\|_{H^2(\mathbb{D})}^2 + 2 \left\| \frac{\varphi_j g(z-1)}{z-1} \right\|_{H^2(\mathbb{D})}^2 \right] \\ &\leq \|f\|_{H^2(\mathbb{D})}^2 + 2|f(1)|^2 \sum_{j=1}^\infty D_1(\varphi_j) + 2\|g\|_{H^2(\mathbb{D})}^2 \\ &\leq 2\|f\|_{D(\delta_1)}^2 + 2|f(1)|^2. \end{aligned}$$

Since $|f(1)| \leq C\|f\|_{D(\delta_1)}$ (see [11]), we conclude that M_Φ is bounded from $D(\delta_1)$ to $\oplus_1^\infty D(\delta_1)$.

(ii): Suppose $\{g_j\}_{j=1}^\infty \subseteq H^2(\mathbb{D})$ such that

$$\varphi_j(z) = \varphi_j(1) + (z-1)g_j(z), \quad \text{and} \quad D_1(\varphi_j) = \|g_j\|_{H^2(\mathbb{D})}^2, \quad j = 1, 2, \dots$$

Note that

$$\begin{aligned} |\varphi_j(z)|^2 &\leq |\varphi_j(1)|^2 + |z-1|^2 |g_j(z)|^2 + 2|\varphi_j(1)||z-1||g_j(z)| \\ &\leq (1+\eta)|\varphi_j(1)|^2 + (1+\frac{1}{\eta})|z-1|^2 |g_j(z)|^2, \end{aligned}$$

where η is any positive number. Then we have

$$\begin{aligned} \epsilon^2 &\leq \sum_{j=1}^{\infty} |\varphi_j(z)|^2 \leq \sum_{j=1}^{\infty} (1+\eta) |\varphi_j(1)|^2 + (1+\frac{1}{\eta}) |z-1|^2 |g_j(z)|^2 \\ &\leq \sum_{j=1}^{\infty} (1+\eta) |\varphi_j(1)|^2 + (1+\frac{1}{\eta}) \frac{|z-1|^2}{1-|z|^2} \sum_{j=1}^{\infty} \|\varphi_j\|_{D(\delta_1)}^2 \\ &\leq \sum_{j=1}^{\infty} (1+\eta) |\varphi_j(1)|^2 + (1+\frac{1}{\eta}) \frac{|z-1|^2}{1-|z|^2} \quad \text{for all } z \in \mathbb{D}, \end{aligned}$$

where in the last inequality we used part (i). Let $z = r \rightarrow 1^-$ we get

$$\epsilon^2 \leq \sum_{j=1}^{\infty} (1+\eta) |\varphi_j(1)|^2 := (1+\eta) |\Phi(1)|^2.$$

Let $\eta \rightarrow 0$, we have $|\Phi(1)|^2 = \sum_{j=1}^{\infty} |\varphi_j(1)|^2 \geq \epsilon^2$, thus $\Phi(1) = (\varphi_1(1), \varphi_2(1), \dots) \neq 0$.

(iii) Suppose $\|M_{\Phi}\| \leq 1$ and $f = \sum_{i=1}^{\infty} [\varphi_i - \varphi_i(1)] \overline{\varphi_i(1)}$, then $f \in H^{\infty}(\mathbb{D})$ and

$$\begin{aligned} \|f\|_{D(\delta_1)}^2 &= \left\| \sum_{i=1}^{\infty} [\varphi_i - \varphi_i(1)] \overline{\varphi_i(1)} \right\|_{D(\delta_1)}^2 \\ &\leq \sum_{i=1}^{\infty} \|\varphi_i - \varphi_i(1)\|_{D(\delta_1)}^2 \sum_{i=1}^{\infty} |\varphi_i(1)|^2 \\ &\leq 2 \left[\sum_{i=1}^{\infty} \|\varphi_i\|_{D(\delta_1)}^2 + \sum_{i=1}^{\infty} |\varphi_i(1)|^2 \right] \sum_{i=1}^{\infty} |\varphi_i(1)|^2 \\ &\leq 4, \end{aligned}$$

where in the last inequality we used part (i).

For any $k \in \mathbb{N}$, let $f_k = \sum_{i=1}^k [\varphi_i - \varphi_i(1)] \overline{\varphi_i(1)}$. Then $f_k \rightarrow f \in D(\delta_1)$, note that $f_k(1) = 0$ and point evaluation at 1 is continuous, we conclude that $f(1) = 0$. \blacksquare

Now we can prove the corona theorem for $M(D(\delta_1))$.

Theorem 3.3. *Let $\{\varphi_j\}_{j=1}^{\infty} \subseteq M(D(\delta_1))$. Suppose that $\|M_{\Phi}\| \leq 1$ and $0 < \epsilon^2 \leq \sum_{j=1}^{\infty} |\varphi_j(z)|^2$ for all $z \in \mathbb{D}$. Then there exists $\{b_j\}_{j=1}^{\infty} \subseteq M(D(\delta_1))$ such that*

- (i) $\Phi(z)B(z)^{\top} = 1$ for all $z \in \mathbb{D}$, and
- (ii) $\|M_B\| \leq \frac{1}{\epsilon} (2 + 8 \frac{C_0}{\epsilon^2} \ln \frac{1}{\epsilon^2})^{1/2}$.

Proof. (i): By Theorem 1.3, there exists an $E \in H_{l^2}^\infty(\mathbb{D})$ such that

$$\Phi(z)E(z)^\top = 1 \quad \text{for } z \in \mathbb{D},$$

and

$$\|E\|_{H_{l^2}^\infty(\mathbb{D})}^2 := \sup_{z \in \mathbb{D}} \sum_{j=1}^{\infty} |e_j(z)|^2 \leq \frac{C_0}{\epsilon^2} \ln \frac{1}{\epsilon^2}.$$

Let $A = \Phi(z)$, $D = E(z)$ in Lemma 3.1, then

$$I - E(z)^\top \Phi(z) = Q_{\Phi(z)} Q_{E(z)}^\top,$$

thus

$$(3.1) \quad I = E(z)^\top \Phi(1) + E(z)^\top (\Phi(z) - \Phi(1)) + Q_{\Phi(z)} Q_{E(z)}^\top.$$

Let $\Phi(1)^* = (\overline{\varphi_1(1)}, \overline{\varphi_2(1)}, \dots)^\top$, then $|\Phi(1)|^2 = \Phi(1)\Phi(1)^*$ and

$$(3.2) \quad \begin{aligned} \Phi(1)^* &= E(z)^\top |\Phi(1)|^2 + E(z)^\top [\Phi(z) - \Phi(1)]\Phi(1)^* \\ &\quad + Q_{\Phi(z)} Q_{E(z)}^\top \Phi(1)^*. \end{aligned}$$

By Lemma 3.2 we have $\Phi(1) = (\varphi_1(1), \varphi_2(1), \dots) \neq 0$, then from (3.2) we have

$$\frac{\Phi(1)^*}{|\Phi(1)|^2} = E(z)^\top + E(z)^\top \frac{[\Phi(z) - \Phi(1)]\Phi(1)^*}{|\Phi(1)|^2} + Q_{\Phi(z)} Q_{E(z)}^\top \frac{\Phi(1)^*}{|\Phi(1)|^2},$$

therefore,

$$\begin{aligned} E(z)^\top + Q_{\Phi(z)} Q_{E(z)}^\top \frac{\Phi(1)^*}{|\Phi(1)|^2} &= \frac{\Phi(1)^*}{|\Phi(1)|^2} - \frac{[\Phi(z) - \Phi(1)]\Phi(1)^*}{|\Phi(1)|^2} E(z)^\top \\ &= \frac{\Phi(1)^*}{|\Phi(1)|^2} - \frac{\sum_{i=1}^{\infty} [\varphi_i(z) - \varphi_i(1)] \overline{\varphi_i(1)}}{|\Phi(1)|^2} E(z)^\top. \end{aligned}$$

Let $B(z)^\top = E(z)^\top + Q_{\Phi(z)} Q_{E(z)}^\top \frac{\Phi(1)^*}{|\Phi(1)|^2}$. From Lemma 3.1, we have

$$\Phi(z)B(z)^\top = 1 \quad \text{for } z \in \mathbb{D},$$

and

$$b_j(z) = \frac{\overline{\varphi_j(1)}}{|\Phi(1)|^2} - \frac{\sum_{i=1}^{\infty} [\varphi_i(z) - \varphi_i(1)] \overline{\varphi_i(1)}}{|\Phi(1)|^2} e_j(z), j = 1, 2, 3, \dots.$$

By Lemma 3.2 we have $f := \sum_{i=1}^{\infty} [\varphi_i - \varphi_i(1)] \overline{\varphi_i(1)} \in M(D(\delta_1))$ and $f(1) = 0$. Thus from Lemma 2.2 we have $b_j \in H^\infty(\mathbb{D}) \cap D(\delta_1) = M(D(\delta_1)), j = 1, 2, \dots$.

(ii): Let $f \in D(\delta_1)$, then

$$\begin{aligned}
& \sum_{j=1}^{\infty} \|b_j f\|_{D(\delta_1)}^2 \\
& \leq \frac{2}{|\Phi(1)|^4} \left[\sum_{j=1}^{\infty} \|\overline{\varphi_j(1)} f\|_{D(\delta_1)}^2 + \sum_{j=1}^{\infty} \left\| \sum_{i=1}^{\infty} [\varphi_i - \varphi_i(1)] \overline{\varphi_i(1)} e_j f \right\|_{D(\delta_1)}^2 \right] \\
& \leq \frac{2}{|\Phi(1)|^4} \left[|\Phi(1)|^2 \|f\|_{D(\delta_1)}^2 + \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \|[\varphi_i - \varphi_i(1)] e_j f\|_{D(\delta_1)}^2 |\Phi(1)|^2 \right] \\
& = \frac{2}{|\Phi(1)|^2} \left[\|f\|_{D(\delta_1)}^2 + \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \|[\varphi_i - \varphi_i(1)] e_j f\|_{D(\delta_1)}^2 \right]
\end{aligned}$$

Note that

$$\begin{aligned}
(3.3) \quad & \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \|[\varphi_i - \varphi_i(1)] e_j f\|_{D(\delta_1)}^2 \\
& = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \|[\varphi_i - \varphi_i(1)] e_j f\|_{H^2(\mathbb{D})}^2 + \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \left\| \frac{\varphi_i - \varphi_i(1)}{z - 1} e_j f \right\|_{H^2(\mathbb{D})}^2 \\
& \leq \|E\|_{H_{l^2}^\infty(\mathbb{D})}^2 \sum_{i=1}^{\infty} \left[\|(\varphi_i - \varphi_i(1)) f\|_{H^2(\mathbb{D})}^2 + \left\| \frac{\varphi_i - \varphi_i(1)}{z - 1} f \right\|_{H^2(\mathbb{D})}^2 \right] \\
& = \|E\|_{H_{l^2}^\infty(\mathbb{D})}^2 \sum_{i=1}^{\infty} \|(\varphi_i - \varphi_i(1)) f\|_{D(\delta_1)}^2 \\
& \leq 2 \|E\|_{H_{l^2}^\infty(\mathbb{D})}^2 \left[\sum_{i=1}^{\infty} \|\varphi_i f\|_{D(\delta_1)}^2 + \sum_{i=1}^{\infty} \|\varphi_i(1) f\|_{D(\delta_1)}^2 \right] \\
& \leq 2 \|E\|_{H_{l^2}^\infty(\mathbb{D})}^2 \left[\|M_\Phi\|^2 + |\Phi(1)|^2 \right] \|f\|_{D(\delta_1)}^2 \\
& \leq 4 \|E\|_{H_{l^2}^\infty(\mathbb{D})}^2 \|f\|_{D(\delta_1)}^2.
\end{aligned}$$

Thus

$$\sum_{j=1}^{\infty} \|b_j f\|_{D(\delta_1)}^2 \leq \frac{2}{|\Phi(1)|^2} \left[\|f\|_{D(\delta_1)}^2 + 4 \|E\|_{H_{l^2}^\infty(\mathbb{D})}^2 \|f\|_{D(\delta_1)}^2 \right],$$

therefore

$$\begin{aligned}\|M_B\| &\leq \left[\frac{2}{|\Phi(1)|^2} (1 + 4\|E\|_{H^\infty(\mathbb{D})}^2) \right]^{1/2} \\ &\leq \frac{1}{\varepsilon} (2 + 8\frac{C_0}{\varepsilon^2} \ln \frac{1}{\varepsilon^2})^{1/2},\end{aligned}$$

where in the last inequality we used $|\Phi(1)| \geq \varepsilon$ in the proof of Lemma 3.2. \blacksquare

Remark 3.4. From equation (3.1), we can get another corona solution $D(z) = (d_1(z), d_2(z), \dots)$ such that

$$(3.4) \quad \sum_{j=1}^{\infty} \varphi_j(z) d_j(z) = 1, \quad z \in \mathbb{D}.$$

Suppose $|\varphi_1(1)| = \max_{\{j=1,2,\dots\}} |\varphi_j(1)|$, let $d_1(z) = \frac{1}{\varphi_1(1)} - \frac{\varphi_1(z) - \varphi_1(1)}{\varphi_1(1)} e_1(z)$, $d_j(z) = -\frac{\varphi_1(z) - \varphi_1(1)}{\varphi_1(1)} e_j(z)$, $j = 2, 3, \dots$. Then (3.4) is satisfied and we have

$$\|M_D\| \leq \left[\frac{2}{|\varphi_1(1)|^2} + 4 \left(\frac{\|\varphi_1\|_{M(D(\delta_1))}^2}{|\varphi_1(1)|^2} + 1 \right) \frac{C_0}{\varepsilon^2} \ln \frac{1}{\varepsilon^2} \right]^{1/2},$$

but in this case the bound of the corona solution depends on the chosen φ_1 . It would be of interested to determine the best possible bound for the solution B in terms of $\|M_\Phi\|$ and ε .

3.2. For general k , we use induction to prove Theorem 1.4.

Proof. The idea is the same as in Theorem 3.3. We sketch a proof here.

If $k = 1$, then by Theorem 3.3, it is true.

Suppose $k = l \geq 1$, it is true.

If $k = l + 1$, note that $\{\varphi_j\}_{j=1}^\infty \subseteq M(D(\mu_{l+1})) \subseteq M(D(\mu_l))$, by induction, there exists $\{e_j\}_{j=1}^\infty \subseteq M(D(\mu_l))$ such that

$$\Phi(z) E(z)^\top = 1 \quad \text{for } z \in \mathbb{D},$$

and

$$\|M_E\| \leq \frac{1}{\varepsilon} \left(2 + 16 \|M_{B_{l-1}}\|^2 \right)^{1/2},$$

Following the same argument as in Lemma 3.2, we have $\Phi(\zeta_{l+1}) = (\varphi_1(\zeta_{l+1}), \varphi_2(\zeta_{l+1}), \dots) \neq 0$ and

$$(3.5) \quad I = E(z)^\top \Phi(\zeta_{l+1}) + E(z)^\top (\Phi(z) - \Phi(\zeta_{l+1})) + Q_{\Phi(z)} Q_{E(z)}^\top.$$

Thus

$$b_j(z) = \frac{\overline{\varphi_j(\zeta_{l+1})}}{|\Phi(\zeta_{l+1})|^2} - \frac{\sum_{i=1}^\infty [\varphi_i(z) - \varphi_i(\zeta_{l+1})] \overline{\varphi_i(\zeta_{l+1})}}{|\Phi(\zeta_{l+1})|^2} e_j(z) \in M(D(\mu_l)),$$

and $\Phi(z)B(z)^\top = 1$ for all $z \in \mathbb{D}$.

Now we estimate $\|M_B\|$. Let $f \in D(\mu_{l+1})$, then

$$\begin{aligned} & \sum_{j=1}^{\infty} \|b_j f\|_{D(\mu_{l+1})}^2 \\ & \leq \frac{2}{|\Phi(\zeta_{l+1})|^2} \left[\|f\|_{D(\mu_{l+1})}^2 + \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \|[\varphi_i - \varphi_i(\zeta_{l+1})]e_j f\|_{D(\mu_{l+1})}^2 \right]. \end{aligned}$$

Suppose $\mu_{l+1} = \mu_l + \delta_{\zeta_{l+1}}$, note that using inequality (3.3) we have

$$\begin{aligned} & \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \|[\varphi_i - \varphi_i(\zeta_{l+1})]e_j f\|_{D(\mu_{l+1})}^2 \\ & \leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \|[\varphi_i - \varphi_i(\zeta_{l+1})]e_j f\|_{D(\mu_l)}^2 \\ & \quad + \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \|[\varphi_i - \varphi_i(\zeta_{l+1})]e_j f\|_{D(\delta_{\zeta_{l+1}})}^2 \\ & \leq \sum_{i=1}^{\infty} \|M_E\|^2 \|[\varphi_i - \varphi_i(\zeta_{l+1})]f\|_{D(\mu_l)}^2 + 4\|E\|_{H_i^\infty(\mathbb{D})}^2 \|f\|_{D(\delta_{\zeta_{l+1}})}^2 \\ & \leq \|M_E\|^2 2 \left[\|M_\Phi\| + |\Phi(\zeta_{l+1})|^2 \right] \|f\|_{D(\mu_{l+1})}^2 + 4\|E\|_{H_i^\infty(\mathbb{D})}^2 \|f\|_{D(\delta_{\zeta_{l+1}})}^2 \\ & \leq 4\|M_E\|^2 \|f\|_{D(\mu_{l+1})}^2 + 4\|M_E\|^2 \|f\|_{D(\mu_{l+1})}^2 \\ & = 8\|M_E\|^2 \|f\|_{D(\mu_{l+1})}^2. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{j=1}^{\infty} \|b_j f\|_{D(\mu_{l+1})}^2 & \leq \frac{2}{|\Phi(\zeta_{l+1})|^2} \left[\|f\|_{D(\mu_{l+1})}^2 + 8\|M_E\|^2 \|f\|_{D(\mu_{l+1})}^2 \right] \\ & \leq \frac{1}{\varepsilon^2} \left(2 + 16\|M_E\|^2 \right) \|f\|_{D(\mu_{l+1})}^2, \end{aligned}$$

and so $\|M_B\| \leq \frac{1}{\varepsilon} \left(2 + 16\|M_E\|^2 \right)^{1/2}$. ■

4. BASS STABLE RANK FOR $M(D(\sum_{i=1}^k a_i \delta_{\zeta_i}))$

The notion of stable rank of a ring was introduced by Bass [1] to facilitate computations in algebraic K-theory. Let us recall the main definition.

Definition 4.1. Let \mathcal{A} be any ring with identity 1. An n -tuple $a = (a_1, \dots, a_n) \in \mathcal{A}^n$ is called unimodular or invertible, if there exists

an n -tuple $b = (b_1, \dots, b_n) \in \mathcal{A}^n$ such that $\sum_{i=1}^n a_i b_i = 1$. The set of all invertible n -tuples is denoted by $U_n(\mathcal{A})$. An $(n+1)$ -tuple $x = (x_1, \dots, x_{n+1}) \in \mathcal{A}^{n+1}$ is called *reducible*, if there exists an n -tuple $y = (y_1, \dots, y_n)$ such that $(x_1 + y_1 x_{n+1}, \dots, x_n + y_n x_{n+1})$ is invertible. The Bass stable rank of \mathcal{A} is the least n such that every invertible $(n+1)$ -tuple is reducible.

In recent years, the Bass stable rank has been studied by many authors in the setting of Banach algebras. Jones, Marshall and Wolff [3] showed that the Bass stable rank of the disc algebra $A(\mathbb{D})$ is one; Treil [12] proved that the Bass stable rank of $H^\infty(\mathbb{D})$ is one; and in [4], Mortini, Sasane, and Wick showed that the Bass stable rank of $\mathbb{C} + BH^\infty$ and A_B are one as well. In this paper, we show that the Bass stable rank of $M(D(\mu_k))$ is also one, where $\mu_k = \sum_{i=1}^k a_i \delta_{\zeta_i}$.

First, we prove that the Bass stable rank of $M(D(\delta_1)) = D(\delta_1) \cap H^\infty(\mathbb{D})$ is one.

Lemma 4.2. *The Bass stable rank of $D(\delta_1) \cap H^\infty(\mathbb{D})$ is one.*

Proof. Let (f, h) be a unimodular pair in $(D(\delta_1) \cap H^\infty(\mathbb{D}))^2$, i.e., there exists $(g_1, g_2) \in (D(\delta_1) \cap H^\infty(\mathbb{D}))^2$ such that $fg_1 + hg_2 = 1$. Then $\inf_{z \in \mathbb{D}} |f(z)| + |h(z)| := \eta > 0$.

Case 1. If $f(1) \neq 0$, then we claim $(f, (f - f(1))h)$ is unimodular.

In fact, if $z \in \mathbb{D}$ is such that $|f(z) - f(1)| \geq \frac{|f(1)|}{2}$, then $|f(z)| + |(f(z) - f(1))h(z)| \geq |f(z)| + \frac{|f(1)|}{2}|h(z)| \geq \min\{1, \frac{|f(1)|}{2}\}\eta$.

If $z \in \mathbb{D}$ is such that $|f(z) - f(1)| \leq \frac{|f(1)|}{2}$, then $|f(z)| = |f(z) - f(1) + f(1)| \geq |f(1)| - |f(z) - f(1)| \geq \frac{|f(1)|}{2}$, and so $|f(z)| + |(f(z) - f(1))h(z)| \geq |f(z)| \geq \frac{|f(1)|}{2}$.

Thus, $(f, (f - f(1))h)$ is unimodular. By Theorem 1 in [12], there is some element $g \in H^\infty(\mathbb{D})$ such that $f + g[(f - f(1))h]$ is invertible in $H^\infty(\mathbb{D})$. Note that $g(f - f(1)) \in D(\delta_1) \cap H^\infty(\mathbb{D})$, by the corona theorem for $M(D(\delta_1))$, we get that $f + g[(f - f(1))h]$ is also invertible in $D(\delta_1) \cap H^\infty(\mathbb{D})$.

Case 2. If $f(1) = 0$, then $h(1) \neq 0$, since $\inf_{z \in \mathbb{D}} |f(z)| + |h(z)| := \eta > 0$. We claim the pair $(f + h, h)$ is unimodular: By the corona theorem for $M(D(\delta_1))$, there exists $(g_1, g_2) \in (D(\delta_1) \cap H^\infty(\mathbb{D}))^2$ such that $fg_1 + hg_2 = 1$, so $(f + h)g_1 + h(g_2 - g_1) = 1$, which implies $(f + h, h)$ is unimodular.

By Case 1, there exists some $g \in D(\delta_1) \cap H^\infty(\mathbb{D})$, such that $(f + h) + gh$ is invertible in $D(\delta_1) \cap H^\infty(\mathbb{D})$. Note that $(f + h) + gh = f + (1 + g)h$, and $1 + g \in D(\delta_1) \cap H^\infty(\mathbb{D})$, we are done. ■

Now we show the Bass stable rank of $M(D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2})) = D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}) \cap H^\infty(\mathbb{D})$ is one.

Lemma 4.3. *The Bass stable rank of $D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}) \cap H^\infty(\mathbb{D})$ is one.*

Proof. Let (f, h) be a unimodular pair in $(D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}) \cap H^\infty(\mathbb{D}))^2$.

Case 1. $f(\zeta_2) \neq 0$. As in Lemma 4.2 we conclude that $(f, (f - f(\zeta_2))h)$ is unimodular. Then by Lemma 4.2, there exists some $g \in D(\delta_1) \cap H^\infty(\mathbb{D})$ such that $f + g[(f - f(\zeta_2))h]$ is invertible in $D(\delta_1) \cap H^\infty(\mathbb{D})$. Note that $g(f - f(\zeta_2)) \in D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}) \cap H^\infty(\mathbb{D})$, by the corona theorem for $M(D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}))$, we get $f + g[(f - f(1))h]$ is also invertible in $D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}) \cap H^\infty(\mathbb{D})$.

Case 2. $f(\zeta_2) = 0$. As in Lemma 4.2, we consider the pair $(f + h, h)$ and conclude that the Bass stable rank of $D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}) \cap H^\infty(\mathbb{D})$ is one. ■

For general k , by induction we obtain that the Bass stable rank of $M(D(\mu_k))$ is one.

Theorem 4.4. *The Bass stable rank of $M(D(\mu_k))$ is one.*

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